ANALYSIS AND DESIGN OF MODEL FOLLOWING CONTROL SYSTEMS BY STATE SPACE TECHNIQUES*

Heinz Erzberger
Research Scientist
Ames Research Center, NASA
Moffett Field, California 94035

Abstract

The problem of designing a model follower control system and of deciding when the plant can follow the model without error is considered. Specifically, a testing procedure is given which determines when the output of a linear plant can be made to follow a model by placing feedback around the plant. It is assumed that model and plant are described by a known set of state equations. The tests are developed for two configurations of model following. In the first of these, known as implicit model following or matching dynamics, the model does not actually become a part of the total system but enters only into the selection of the feedback law, whereas in the second or real model following configuration, simulation of the model is required. It is shown that in the absence of disturbances the conditions for perfect following are essentially the same for both configurations. For the case of implicit model following, the control law which achieves perfect following is also calculated and, in general, will require both finite and singular controls of the delta function type if the model state space is of smaller dimension than the plant state space. The results obtained suggest a rational guide for deciding what type of model following is most appropriate for a given problem. That is, real model following, at the expense of greater complexity, offers the best performance if random disturbances occur within the plant but achieves no better performance than the simpler implicit model following in the absence of disturbances or uncertainties. Two examples, one of which is based on the lateral equation of motion of an aircraft, are given to illustrate the theory.

Introduction

The design of a model follower control system consists of choosing a feedback law so that the output variables of the plant will faithfully follow the output variables of a model. For instance, the plant could be represented by the linearized equations of motion of an aircraft and the model by the equations corresponding to an aircraft with ideal response characteristics as determined perhaps from simulator studies. This paper is not concerned with the methods of deriving an adequate model but assumes at the outset that one is given as part of the problem.

Recently Ellert and Merriam [1] and Tyler [2] used quadratic optimal control theory to synthesize model following control systems. Their technique, unlike those based on classical procedures, is applicable to arbitrary multivariable systems and always yields a feedback configuration which minimizes a quadratic function of the error between the plant output and the model.

Although quadratic optimal control yields a general procedure for synthesizing model following control systems, experience has brought to light some additional design problems which can be handled more effectively by other methods. An example of such a problem is that of deciding when the closed-loop plant can follow the model perfectly. That is, for a particular combination of plant and model, one may find that the closed-loop plant designed by the methods of optimal control follows the model with unacceptably large errors which cannot be reduced below some limiting value merely by manipulating the weight matrices in the cost function. In other words, there may not be enough freedom of choice in the feedback matrix to match the plant to the model if the model dynamics differ greatly from the open-loop plant dynamics. In that case, the feedback matrix calculated via optimal control still yields a least squares match between model and plant response during the control period but gives no prior indication of matching accuracy which must be determined separately either by actually checking the response of the closed loop system or by evaluating the minimum cost. A simple test of whether a plant and model can be matched when no restrictions are placed on the magnitude of the control signals would therefore be a useful aid in designing a model following control system.

A further difficulty occurs when faced with the choice between a design based on a "model in the system" and the so-called "model in the performance index" or implicit model. The relationship between these two design approaches is not entirely clarified in the literature, although Tyler [2] has shed some light on this question.

*October 18, 1967

The main subject of this paper is to develop simple algebraic tests, applicable to both types of model following designs, for checking whether a plant can be matched perfectly to a model. The tests are of such a nature that they also reveal the necessity of an unbounded control law if required for perfect matching. For implicit model following, if the control law which achieves matching is known to exist, it will also be derived.

Implicit Model Following

In implicit model following one attempts to modify the output dynamics of the plant by means of feedback so as to approximate the dynamics of a given model. Mathematically, implicit model following is defined in the following way. Let the multivariable plant be described by the equations

$$\dot{x} = Fx + Bu \tag{1}$$

$$y = Hx \tag{2}$$

where x is an n-dimensional state vector, u is an m-dimensional control vector, and y is an l-dimensional output vector. The matrices F, B, and H do not depend on time and have dimensions $n \times n$, $n \times m$, and $l \times n$, respectively. Also, it is assumed that $n \geq m$ and $n \geq l$. If the model is described by the equation

$$\dot{z} = Lz \qquad L = l \times l \text{ constant matrix} \tag{3}$$

where z denotes the l-dimensional state vector of the model, then we ask that the output $y(t)$ satisfy

$$\frac{dy(t)}{dt} = Ly(t) \tag{4}$$

as closely as possible. One method of achieving this objective is to use optimal control theory to minimize the following quadratic loss function [3]:

$$I = \int_0^\infty [(\dot{y} - Ly)'Q(\dot{y} - Ly) + u'Ru]dt$$

where Q is a positive semidefinite matrix and R is a positive definite matrix. This formulation of model following does not introduce the state variables of the model directly, since y and $\dot{y}$ appearing in the loss function can be expressed as a function of x and u. Hence, the names "implicit model," "model in the performance index" [2], and "matching dynamics" [4] all have been used to describe this method. Although optimal control theory generates a feedback matrix for arbitrary L, the error $(\dot{y} - Ly)$ obtained from the closed-loop system may be large even if the norm of R is chosen very small. Furthermore, one can show that assumptions such as controllability of the plant or observability of the output also are neither necessary nor sufficient to guarantee that the error will be small.

We now address ourselves to the question, "Under what conditions is it possible to satisfy equation (4) exactly?" Using equations (1) and (2) and requiring that equation (4) is a strict equality permits us to write

$$HBu = (LH - HF)x \tag{5}$$

If this equation is to hold for all x by proper choice of u, then the range of HB must contain the range of $(LH - HF)$ written as

$$\mathcal{R}(HB) \supset \mathcal{R}(LH - HF) \tag{6}$$

First, equation (5) is formally solved for u by taking the pseudoinverse of HB [5]

$$u = (HB)^\dagger(LH - HF)x \tag{7}$$

Then, when u is eliminated from equations (5) and (7), the condition for zero error becomes

$$((HB)(HB)^\dagger - I)(LH - HF)x = 0 \qquad \text{all } x \tag{8}$$

To justify the use of the pseudoinverse, one must show that if equation (8) is true for all x (i.e., $((HB)(HB)^\dagger - I)(LH - HF)$ is the zero transformation), then relation (6) is a necessary consequence. A property of the pseudoinverse which permits this conclusion is that $(HB)(HB)^\dagger$ is an orthogonal projection operator on $\mathcal{R}(HB)$. Let z be any vector in $\mathcal{R}(LH - HF)$ and write z as the sum

$$z = z_0 + z_\perp$$

where

$$z_0 \in \mathcal{R}(HB) \qquad \text{and} \qquad z_\perp \in [\mathcal{R}(HB)]^\perp$$

Since $((HB)(HB)^\dagger - 1)$ is also an orthogonal projection which projects every $z \in \mathcal{R}(LH - HF)$ on $[\mathcal{R}(HB)]^\perp$ and since by assumption equation (8) is zero for every x, it follows that

$$z = z_0$$

and therefore $\mathcal{R}(LH - HF) \subset \mathcal{R}(HB)$. Thus, we conclude that choosing

$$u = (HB)^\dagger(LH - HF)x \tag{9}$$

when

$$((HB)(HB)^\dagger - I)(LH - HF) = 0 \tag{10}$$

guarantees that $\dot{y} = Ly$, or, equivalently, that the output dynamics of the closed-loop system will match the desired output dynamics. Furthermore, the boundedness of the pseudoinverse implies that the feedback law $(HB)^\dagger(LH - HF)$ is bounded. Therefore, if the condition for zero error (eq. (10)) is satisfied and the model is stable, the controls that achieve a perfect match are always bounded.

When equation (10) is not satisfied, it may still be possible to achieve zero error by enlarging the class of controls to include delta functions. As the next step the control law and the test for perfect following derived above are extended to the case of unbounded controls. One begins by writing every control as a sum of ordinary and delta functions:

$$u(t,\tau) = u_1(t) + u_\delta(\tau)\delta(t - \tau) \tag{11}$$

$$u_1 \in [\mathfrak{N}(HB)]^\perp$$

$$u_\delta \in \mathfrak{N}(HB)$$

where $\mathfrak{N}$ denotes null space, $\perp$ denotes the perpendicular complement, τ is a running variable, and t is current time assumed to be fixed. Thus the delta function occurs at time t. It is necessary to restrict u_δ to the null space of HB since otherwise the left side of equation (5) would contain a delta function of strength HBu_δ while the right side does not. Hence, perfect matching would be absent at the moment the impulse occurs. From this remark it also follows that delta function controls are only helpful if the rank of HB is less than maximal. Adding the proposed delta function control at time t^+ can then be shown to modify the derivative of the output, $\dot{y}(t^+)$, as follows

$$\dot{y}(t^+) = HFx + HBu_1 + HFBu_\delta \tag{12}$$

Since the right side of equation (12) must be equal to Ly if zero error is to be achieved, one obtains

$$HBu_1 + HFBu_\delta = (LH - HF)x \tag{13}$$

By defining $\tilde{u} = u_1 + u_\delta$ and using the pseudoinverse to construct the necessary projections, one can write

$$u_1 = (HB)^\dagger(HB)\tilde{u}\ , \qquad u_\delta = (1-(HB)^\dagger HB)\tilde{u} \tag{14}$$

Then, upon substituting equations (14) into equation (13), it is possible to solve explicitly for $\tilde{u}$:

$$\tilde{u} = M^\dagger(LH - HF)x \tag{15}$$

where

$$M = HB + HFB(1 - (HB)^\dagger HB)$$

Finally, the condition for zero error can now be derived by replacing u_1 and u_δ in equation (13) with the relationship for these quantities obtained from equations (14) and (15):

$$(MM^\dagger - I)(LH - HF) = 0 \tag{16}$$

If condition (16) is satisfied, then equation (15) essentially gives the control law which achieves zero error, except for the implementation of the delta function control of that component of $\tilde{u}$ which lies along $\mathfrak{N}(HB)$. Assuming for the moment that it is possible to generate the required delta function, we want to demonstrate that from t^+ onward, equality of equation (4) can be maintained. In general, equation (4) or (5) will not hold at time t since the effect of the delta function is not felt until time t^+. At that moment a step change occurs in $\dot{y}$ in such a way that equation (4) is satisfied. Perfect matching is, therefore, assured for at least a time interval that is short in comparison to the fastest time constant of the system. As soon as the difference between $\dot{y}$ and Ly exceeds some small threshold, where the value of the threshold may be chosen arbitrarily small, another delta function whose weight is chosen according to equations (14) and (15) is applied. The second delta function restores the equality of equation (4). Clearly, perfect matching can thus be maintained indefinitely by continuing to apply a delta function whenever the threshold value is exceeded. We also note that the smaller the threshold value is chosen, the closer will be the spacing of the delta functions, but also the smaller will be their strength.

The problem of implementing a closed-loop control law which generates the required delta functions is discussed in the appendix. It is shown there that an approximate synthesis of such a control law is obtained by multiplying u_δ by a large positive gain constant K and that the approximation to the ideal delta function control law improves in proportion to the magnitude of K.

If equation (16) is not satisfied and the rank of M is not yet maximal, it may still be possible to achieve zero error by including various derivatives of delta functions in addition to the previously used controls. Consider, for example, the addition of first derivatives of delta functions. The control u is then written as the direct sum of three controls:

$$u = u_1 + u_\delta \delta(t - \tau) + u_{\delta 1}\delta^1(t - \tau) \quad (17)$$

$$u_1 \in [\mathfrak{N}(HB)]^\perp$$

$$u_\delta \in \mathfrak{N}(HB) \bigcap [\mathfrak{N}(HFB)]^\perp$$

$$u_{\delta 1} \in \mathfrak{N}(HB) \bigcap \mathfrak{N}(HFB)$$

This decomposition assures that delta functions and their derivatives do not appear in the expression for $\dot{y}$ which, for the choice of controls given by equation (17) becomes

$$\dot{y} = HFx + HBu_1 + HFBu_\delta + HF^2Bu_{\delta 1}$$

At this point, the right side of the above equation is equated to LHx and the condition describing when the resulting equation has a solution u for all x is found. As before, a numerical test for perfect matching similar to equation (16) can be developed by defining $\tilde{u} = u_1 + u_\delta + u_{\delta 1}$ and then writing the components of $\tilde{u}$ as orthogonal projections of $\tilde{u}$ on the appropriate subspaces:

$$u_1 = P_1\tilde{u} , \quad u_\delta = P_\delta\tilde{u} , \quad u_{\delta 1} = P_{\delta 1}\tilde{u}$$

For general vector controls which contain finite, delta function, and derivative of delta function components it does not appear to be possible to construct the projections explicitly in terms of pseudoinverses, although numerical procedures for performing such constructions are well known. The condition for perfect matching and the corresponding control law for this case is still given by equations (16) and (15), respectively, but M must now be replaced by

$$\hat{M} = HB + HFBP_\delta + HF^2BP_{\delta 1} \quad (18)$$

Real Model Following

In implicit model following design the model entered only in the selection of the feedback matrix, and no real time error measurement between model output and plant output was necessary. In real model following the model, although itself uncontrollable, becomes part of the system in that the model states are compared with the output of the plant. The comparison actually takes place in the performance measure as follows:

$$I = \int_0^\infty [(y - z)'Q(y - z) + u'Ru]dt \quad (19)$$

If optimal control theory is to be used to compute the feedback and feedforward gain matrices, one augments the state space of the plant with the model states and then minimizes

$$I = \int_0^\infty (w'\hat{Q}w + u'Ru)dt \quad (20)$$

where

$$\dot{w} \equiv \begin{matrix} \dot{x} \\ \hline \dot{z} \end{matrix} = \begin{bmatrix} F & 0 \\ 0 & L \end{bmatrix} w , \quad \hat{Q} = \begin{bmatrix} H'QH & -H'Q \\ -QH & Q \end{bmatrix}$$

It has been shown that the feedback matrix computed by this method depends only on the F, Q, and R and not on the model parameters L, whereas the feedforward matrix depends on both model and plant parameters [2].

The derivation of conditions for perfect following in this case uses the fact that if all orders of time derivatives of the error are zero at any time t, then the error will be zero for all time. Beginning with the zeroth derivative one obtains the obvious fact that

$$z(t) = Hx(t) \quad (21)$$

If the first derivative of the error is to be zero, one finds after using equations (1), (2), (3), and (21) that

$$H(Fx + Bu + BAz) = H(Fx + Bu + BAHx) = LHx \quad (22)$$

Here A is assumed to be an arbitrary feedforward matrix. Upon solving the last two members of equation (22) for u, one obtains

$$u = (HB)^\dagger(LH - HF - HBAH)x \quad (23)$$

This control law achieves zero error in the first derivative if

$$((HB)(HB)^\dagger - I)(LH - HF) = 0 \quad (24)$$

which is obtained by substituting u of equation (23) into equation (22). But equation (24), if true, holds for arbitrary t; therefore, all higher order derivatives of the error will also be zero. Thus, the condition for perfect following for this case is the same as for implicit model following; moreover, direct use of the model states $z(t)$ through the feedforward loop has no effect on this condition.

If condition (24) is not satisfied, then one can consider sums of ordinary and delta function controls as a means of achieving zero error, just as for the case of implicit model following. Clearly, the arguments presented there carry over to this case.

Evaluation of Real and Implicit Model Following

It has been shown that the conditions for perfect following and the control law that achieves perfect following are identical for both real and implicit model following. Thus, assuming that perfect following is possible with either bounded or unbounded controls and that unknown disturbances are absent, there is no essential advantage of one design over the other. The key issue in deciding between a real model following design (with its additional hardware requirements) and the simpler implicit model following is whether or not the requirements of the problem dictate that a particular phase trajectory of the model be followed in the presence of unknown disturbances in the plant. Implicit model following is not capable of following a phase trajectory of the model where disturbances are present since no real time error measurement between model and plant states takes place; the model following is open loop as it were. But, if the model serves merely to characterize the desired dynamic properties of the plant, in other words model and plant should have similar responses when starting at the same initial states with no disturbances present, the implicit model following would be sufficient.

The maintenance of alinement between plant and model in the presence of uncertainties, be they unknown parameters or random disturbances necessitates the use of a real model in the system. With a model in the system, errors arising between model and plant states due to uncertainties can be measured and corrected continuously. Thus, the principal advantage of having a model in the system is not that it always achieves better following, but that it desensitizes the following to unknown disturbances.

In the case of real model following, the control law given by equation (23) cannot be used by itself since it does not include the states of the model. That is, this control law fails to take advantage of the possibility, unique to real model following, of realining the plant and the model states if disturbances cause them to drift apart. Here the techniques of optimal control would seem most appropriate for computing the control law.

Examples

In this section two examples are presented. The first, which is discussed in some detail, represents the linearized lateral equations of motion of an aircraft. Three model following designs, one calculated by the theory developed in this paper, and the other two by the methods of quadratic optimal control, are compared in this example, and the advantages of each are pointed out. The second example illustrates the theory when an unbounded control law is required for perfect matching. All computations were performed with the automatic synthesis program of Kalman and Englar [4].

Example 1

The numerical values for the model and plant parameters used here correspond with one of Tyler's examples [2].

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -2.93 & -4.75 & -0.78 \\ 0.086 & 0 & -0.11 & -1.0 \\ 0 & -0.042 & 2.59 & -0.39 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & -3.91 \\ 0.035 & 0 \\ -2.53 & 0.31 \end{bmatrix}$$

$$H = I$$

$$L = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & -73.14 & 3.18 \\ 0.086 & 0 & -0.11 & -1.0 \\ 0.0086 & 0.086 & 8.95 & -0.49 \end{bmatrix}$$

$$\text{State vector} = \begin{bmatrix} \varphi \\ \dot{\varphi} \\ \beta \\ r \end{bmatrix} \begin{array}{l} \text{(bank angle)} \\ \text{(bank rate)} \\ \text{(sideslip angle)} \\ \text{(yaw rate)} \end{array}$$

$$\text{Control vector} = \begin{bmatrix} \delta_r \\ \delta_a \end{bmatrix} \begin{array}{l} \text{(rudder deflection)} \\ \text{(aileron deflection)} \end{array}$$

The test of perfect following, equation (8), applied to this example gives the following result:

$$((HB)(HB)^{\dagger} - I)(LH - HF) =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1.3\times10^{-7} & -4.3\times10^{-6} & -1.1\times10^{-5} & -3.4\times10^{-6} \\ -1.2\times10^{-4} & -3.9\times10^{-3} & -1.3\times10^{-2} & -3.0\times10^{-3} \\ -1.6\times10^{-6} & -5.4\times10^{-5} & -1.8\times10^{-4} & -4.0\times10^{-5} \end{bmatrix} \tag{25}$$

Thus perfect following is not possible because the right side of the above equation is not the zero transformation. Since HB has maximum rank, it also follows from earlier work that delta function controls cannot improve this situation. Nevertheless, because most entries in the matrix of equation (25) are small in comparison with entries in the system and model matrices, it is interesting to compare the performance of the simple model following control law of equation (7); that is,

$$u = (HB)^{\dagger}(LH - HF)x = \begin{bmatrix} -3.4\times10^{-3} & -0.11 & -0.37 & -0.084 \\ 0 & -0.49 & 17.5 & -1.01 \end{bmatrix} x \tag{26}$$

with those calculated by optimal control for both the implicit and real model following performance indices. The Q, R feedback and feedforward matrices used in the calculation for the implicit and real model following designs are given below:

Implicit Model Following

Diag Q = [0,6,0,6] , Diag R = [1,1]

Feedback matrix =

$$\begin{bmatrix} 0.0034 & 0.111 & 0.371 & 0.0356 \\ 0 & 0.494 & -17.5 & 0.614 \end{bmatrix}$$

Real Model Following

Diag Q = [10,10,10,10] , Diag R = [1,1]

Feedback matrix =

$$\begin{bmatrix} -0.074 & -0.094 & 2.34 & -3.23 \\ -3.15 & -2.73 & 0.835 & 0.261 \end{bmatrix}$$

Feedforward matrix =

$$\begin{bmatrix} 0.031 & -0.246 & 2.109 & -4.62 \\ -3.02 & -3.16 & 16.6 & -2.3 \end{bmatrix}$$

Figures 1 and 2 compare the transient responses of the three different control laws for two initial conditions corresponding to an initial bank angle and an initial bank rate, but with the model and plant states alined at the start. Because the response of the implicit model following law calculated with optimal control was generally not much different from the response obtained by using the control law equation (26), it is not drawn in all the figures in order to reduce crowding of the curves. Also, those state variable time histories that were omitted were found to be as well matched as φ in figure 1(a). It can be seen in figures 1 and 2 that at least during the first 5 seconds, the performance of the control law given by equation (26) compares favorably with both real and implicit model following designed via optimal control. When model and plant are not too dissimilar, as shown in this case by the results of the perfect following test, one can expect this control law to work quite well; but for greatly mismatched model and plant, again as determined by the perfect following test, no assurance of satisfactory operation can be given.

Figures 3(a) and 3(b) demonstrate when it is advantageous to use real model following. Here a disturbance in the plant is assumed to have caused a sudden misalinement between the model and plant bank angle variables. Under this condition, the real time error measurement between model and plant, which is only possible with real model following, facilitates the eventual realinement of corresponding state variables. Thus, the model serves as a memory of a particular trajectory in the presence of disturbances.

<u>Example 2</u>

The open loop plant equations of this example are again fourth order but the model equations are now of second order.

$$\dot{x} = \begin{bmatrix} 0 & 1.0 & 0 & 0 \\ 0 & -2.93 & -4.75 & 0 \\ 0.086 & 0 & -0.11 & -1.0 \\ 0 & -0.042 & 2.59 & -0.39 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u ,$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x , \qquad \dot{z} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} z$$

It can be shown that the test for perfect following with finite controls, equation (10), applied to this example fails; therefore, it is necessary to use the more general test given by equation (16), which considers sums of finite and delta function controls. This latter test shows that perfect following is indeed possible with a control law containing both finite and delta function controls.

After the required calculations are performed, the two parts of the control law are found to be

$$u_1 = \begin{pmatrix} 0 & 1.465 & 2.875 & 0 \\ 0 & 1.465 & 2.875 & 0 \end{pmatrix} x ,$$

$$u_\delta = \begin{pmatrix} -0.086 & -2.0 & -1.89 & 1 \\ 0.086 & 2.0 & 1.89 & -1 \end{pmatrix} x$$

As shown in the appendix, an approximate synthesis of a control law containing delta functions is obtained by multiplying u_δ by a large positive constant K. The total control is then given by the sum of the two components, with K appearing as a parameter in the feedback matrix.

$$u = \begin{bmatrix} -0.086K & 1.465-2K & 2.875-1.89K & K \\ 0.086K & 1.465+2K & 2.875+1.89K & -K \end{bmatrix} x$$

Figures 4(a) and 4(b) demonstrate the convergence properties of the control law as a function of the gain constant K.

Conclusion

Two basic approaches, each having its particular advantages and disadvantages, exist for designing a model follower control system. In the implicit model following method the model enters only into the selection of the feedback law placed around the plant but does not become physically part of the total system. Thus, feedback is used chiefly to modify the dynamics of the plant so that its output behavior coincides with that of the model. This type of following therefore operates open loop with respect to the model since during the control interval no real-time comparison of model states and plant output takes place. The main advantage of this method is simplicity and low cost of implementation because the model in the system need not be simulated.

If design specifications require that the model follower control system be able to follow a specific phase trajectory of the model starting at a given initial state while the plant is subject to unknown disturbances or parameter changes, then real model following is the appropriate choice. Here the continuous measurement of error between model states and plant output offers the additional freedom of using this error, appropriately weighted, as a means of alining the model and the plant. However, because the conditions for perfect following are identical for both real and implicit model following, this additional freedom does not contribute to improved matching of the dynamics of model and plant in comparison with implicit model following.

Although optimal control theory offers the most general method available for the design of model following systems, it is inefficient, because of the computational effort required, for answering such preliminary design questions as whether or not it is possible to match model and plant and whether bounded or unbounded controls are required. The theory presented here answers such questions directly by means of an algebraic test and, in addition, furnishes, for the case of implicit model following, a simply computed control law that achieves perfect matching if the test shows this to be possible. By means of an example it is demonstrated that even if perfect matching is not possible the performance of a system using the simple control law may compare favorably with the performance of systems designed via optimal control as long as the dynamics of plant and model are not too dissimilar.

Appendix

Approximate Synthesis of Feedback Law Containing Delta Functions

In the derivation of conditions for perfect following it was necessary to include delta functions as permissible controls in order to achieve perfect following. Whenever the model is of lower order than the plant, it may be expeditious to sacrifice a part of the plant dynamics for better matching, and in that case delta function controls are necessary. Important questions arise now as to the procedure for constructing a control law containing delta functions and how to approximate one to arbitrary accuracy. Although the concept behind the construction of such a control law is well known, its adaptation to this problem requires some explanation.

To begin with, it is assumed that perfect matching in the implicit model following sense can be achieved with controls containing finite, delta function and derivative of delta function components. These three components of u are given in the main text and are repeated here for convenience:

$$u_1 = P_1\tilde{u} \ , \quad u_\delta = P_\delta\tilde{u} \ , \quad u_{\delta 1} = P_{\delta 1}\tilde{u} \tag{A1}$$

where $\tilde{u}$ is defined as follows

$$\begin{aligned}\tilde{u} &= \hat{M}^\dagger(LH - HF)x \\ &\equiv (HB + HFBP_\delta + HF^2BP_{\delta 1})(LH - HF)x \end{aligned} \tag{A2}$$

The next step is to divide time into equal increments Δt, which are chosen much shorter than the shortest time constant of the model and the plant. The control applied to the plant remains constant throughout each time increment and is updated only at the beginning of a new increment. Assuming the control process starts at $t = 0$, the first control applied to the plant is chosen as follows:

$$u(o) = \left[P_1 + \frac{1}{\Delta t} P_\delta + \frac{2}{(\Delta t)^2} P_{\delta 1}\right]\hat{M}^\dagger(LH - HF)x(o) \tag{A3}$$

We note that the gain constants multiplying those components of control that require delta function and derivative of delta function are $1/\Delta t$ and $2/(\Delta t)^2$, respectively. Also, $x(o)$ is arbitrary and therefore may be such that $\dot{y}(o) \neq Ly(o)$.

This sets the stage for the crucial step of this approach, namely the computation of the error between $\dot{y}$ and Ly at the end of the first time increment. Using the standard expression for the time response of a linear system [5], we compute $y(\Delta t)$:

$$y(\Delta t) = H\left\{e^{F\Delta t}x(o) + \int_0^{\Delta t} e^{F(\Delta t - \tau)}Bu(\tau)d\tau\right\} \tag{A4}$$

where the transition matrix e^{Ft} is given in terms of the infinite series as follows:

$$e^{Ft} = I + Ft + \frac{F^2t^2}{2!} + \ . \ . \ . \tag{A5}$$

Upon substitution of equation (A5) into (A4) and since $u(t)$ is constant within the integration interval, $y(\Delta t)$ becomes

$$\begin{aligned} y(\Delta t) = H\Big\{&x(o) + \Delta tFx(o) + \frac{\Delta t^2}{2!}F^2x(o) + \ . \ . \ . \\ &+ \left[I\Delta t + \frac{F\Delta t^2}{2} + \frac{F^2\Delta t^3}{3!} + \ . \ . \ .\right]Bu(o)\Big\} \end{aligned} \tag{A6}$$

The purpose of calculating $y(\Delta t)$ is to evaluate the error between $\dot{y}(\Delta t)$ and $Ly(\Delta t)$ and to show that it can be made arbitrarily small. The error, denoted by $\eta(\Delta t)$, is evaluated using equations (1), (2), (A3), and (A6):

$$\begin{aligned} &\eta(\Delta t) \\ &= H\Big\{F + \Delta tF^2 + \frac{\Delta t^2F^3}{2!} + \ . \ . \ . + \Big[F\Delta t + \frac{F^2\Delta t^2}{2} \\ &\quad + \frac{F^3\Delta t^3}{3!} + \ . \ . \ .\Big]\Big[BP_1 + \frac{BP_\delta}{\Delta t} \\ &\quad + \frac{2BP_{\delta 1}}{(\Delta t)^2}\Big]\hat{M}^\dagger(LH - HF) + B\Big[P_1 + \frac{1}{\Delta t}P_\delta \\ &\quad + \frac{2}{(\Delta t)^2}P_{\delta 1}\Big]\hat{M}^\dagger(LH - HF)\Big\}x(o) \\ &\quad - \Big\{LH + \Delta tLHF + \frac{\Delta t^2}{2!}LHF^2 + \ . \ . \ . \\ &\quad + LH\Big[I\Delta t + \frac{F\Delta t^2}{2!} + \frac{F^2\Delta t^3}{3!}\Big]\Big[BP_1 \\ &\quad + \frac{BP_\delta}{\Delta t} + \frac{2BP_{\delta 1}}{(\Delta t)^2}\Big]\hat{M}^\dagger(LH - HF)\Big\}x(o) \end{aligned} \tag{A8}$$

Since $HBP_\delta = 0$, $HBP_{\delta 1} = 0$, and $HFBP_{\delta 1} = 0$, equation (A8) simplifies to the following form:

$$\eta(\Delta t) = (\hat{M}\hat{M}^\dagger - I)(LH - HF) + o(\Delta t) \tag{A9}$$

where $o(\Delta t)$ are terms that go to zero at least as fast as Δt. But the first term is identically zero by the assumption that the perfect matching condition, equation (16), is satisfied.

Thus, if Δt is chosen sufficiently small (or the gain constants arbitrarily large), the error at the end of Δt seconds can be made as small as desired for any initial condition.

Furthermore, the error can be maintained arbitrarily small for all future time if the control, equation (A3), is updated at the beginning of each new time increment by replacing $x(n\Delta t)$ with $x[(n + 1)\Delta t]$, n being the number of time increments.

Through appropriate limiting arguments, the discussion given here for a discrete time control law can be suitably generalized to continuous time control.

References

[1] Ellert, F. J., and Merriam, C. W., III, Synthesis of Feedback Controls Using Optimization Theory - An Example. IEEE Trans. Automatic Control, vol. AC-8, no. 2, April 1963, pp. 89-103.

[2] Tyler, J. S., Jr., The Characteristics of Model-Following Systems as Synthesized by Optimal Control. IEEE Trans. Automatic Control, vol. 9, no. 4, Oct. 1964, pp. 485-498.

[3] Kalman, R. E., Contributions to the Theory of Optimal Control. Boletin de la Sociedad Matematica Mexicana, vol. 5, 1960, pp. 102-119.

[4] Kalman, R. E., and Englar, T. S., A User's Manual for Automatic Synthesis Program (Program C). NASA CR-475, 1966.

[5] Zadeh, L. A., and Desoer, C. A., Linear System Theory. Ch. 11 and appendix C. McGraw-Hill, New York, 1963.

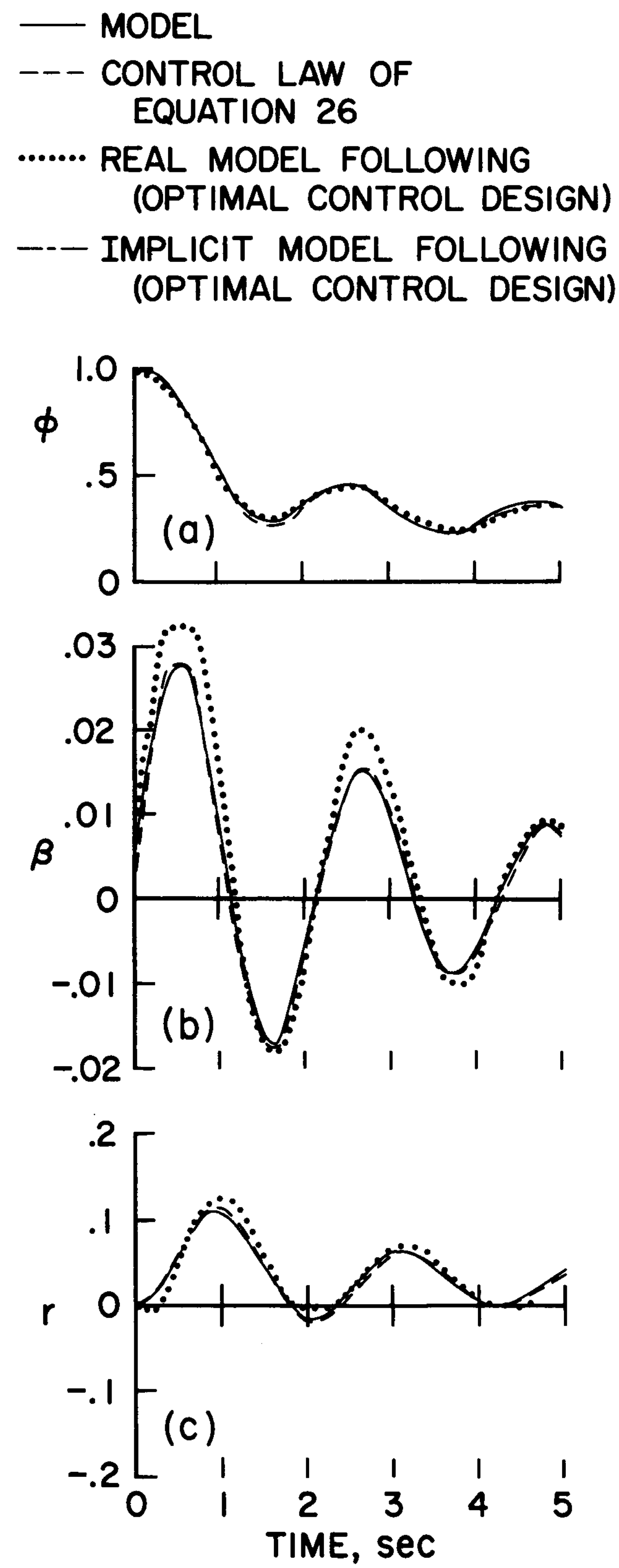

Figure 1.- Transient responses pertaining to example 1: initial bank angle disturbances.

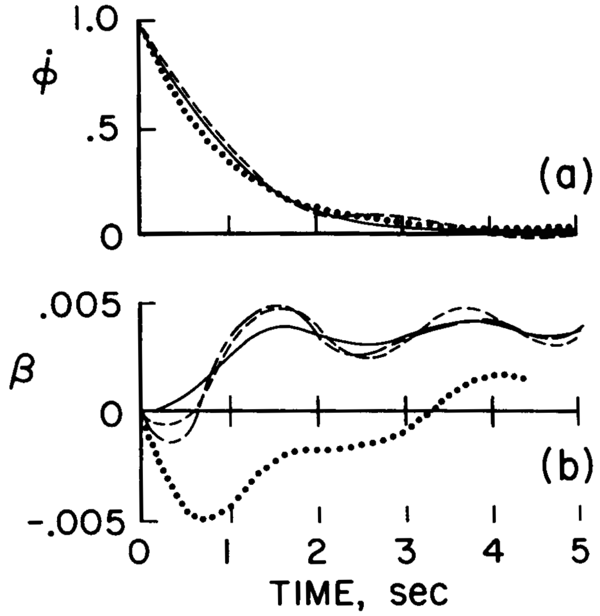

Figure 2.- Transient responses pertaining to example 1: initial roll rate disturbance.

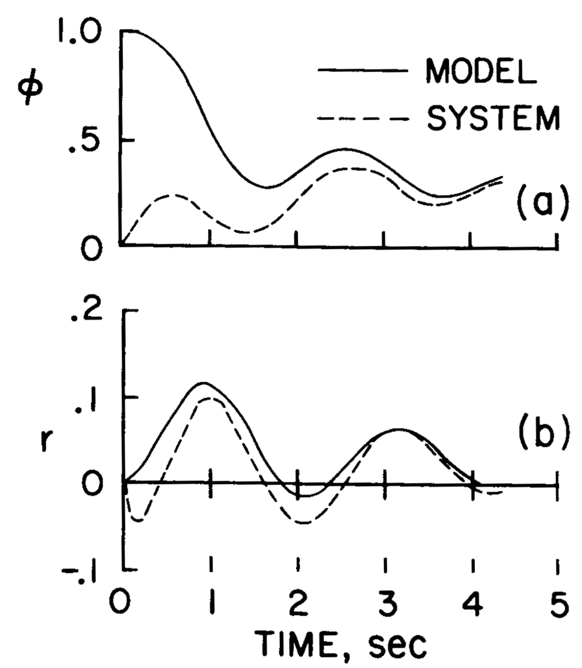

Figure 3.- The effect of initial bank misalinement between model and plant in real model following.

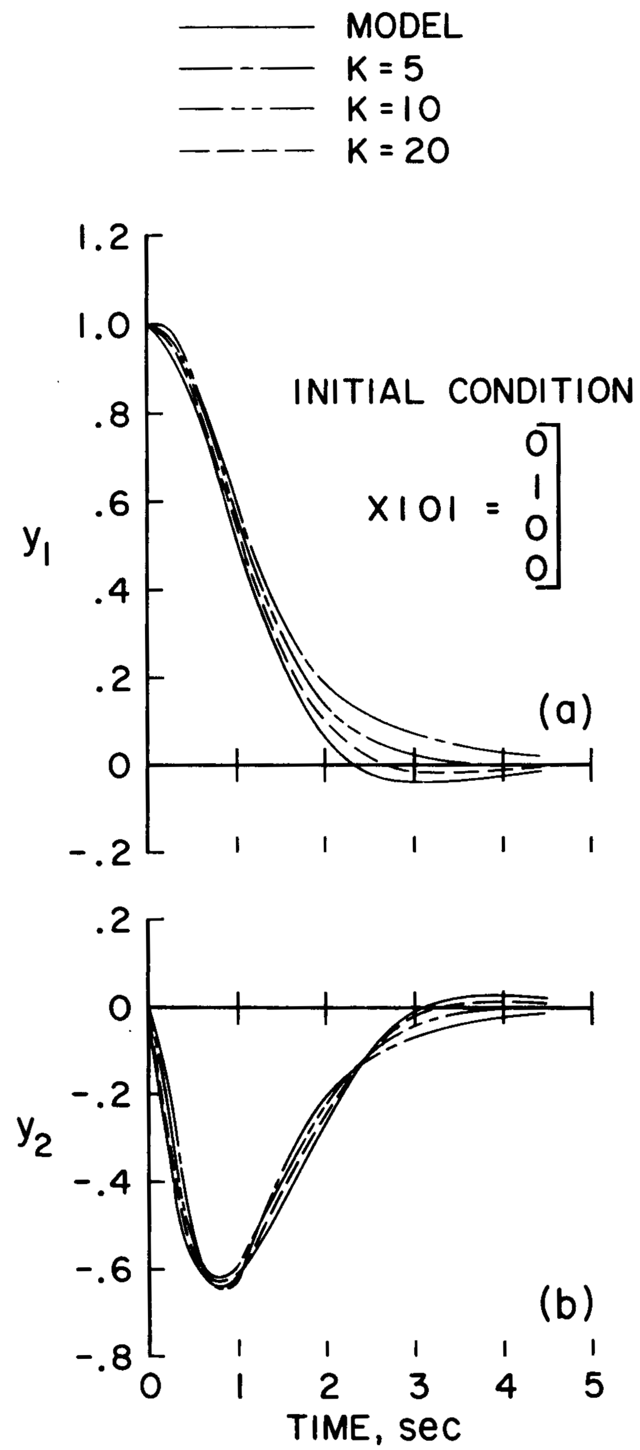

Figure 4.- The effect of K on the transient response of example 2.